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Difference equations and highest-weight modules of $U_q[sl(n)]$

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Abstract. Based on the trigonometric R-matrix of $U_q[sl(2)]$ a family of q-deformed discrete Knizhnik–Zamolodchikov equations is formulated. Solutions to them are constructed by using the generalization of the algebraic Bethe ansatz developed to solve difference equations. These solutions are shown to be of highest weight with respect to the underlying quantum group structure. Using the variant of the nested Bethe ansatz method these results are extended to the higher-ranked symmetry of $U_q[sl(n)]$.

1. Introduction

This paper can be considered as an addendum to a previous paper [1] on matrix difference equations and a generalized version of the Bethe ansatz. For an introduction to their role in mathematical physics the reader is referred to [1–4] and references therein.

One important application of such difference equations is that of so-called form factors in integrable quantum field theories as done recently in [2] for the Sine–Gordon model.

Though q-deformations of discrete Knizhnik–Zamolodchikov equations have been treated in much detail in recent years [4, 5], it is not completely clear how those solutions are related to the underlying symmetry of such problems.

The conventional algebraic formulation of the Bethe ansatz demonstrates the close relation between the eigenvector problem and the representation theory of its connected symmetry group (either classical or q-deformed): Bethe vectors can be constructed as highest weight vectors of irreducible representations and therefore by simply counting them one makes certain of spanning the whole space of states.

However, one has to be careful when moving from classical Lie algebras to a quantum group, as can be seen when a one-dimensional periodic XXX Heisenberg chain is deformed to the anisotropic XXZ model. Deforming the Hamiltonian in a straightforward way will not preserve the (quantum) symmetry. Instead one is forced to change the boundary conditions [6] or to take additional terms (arising from the nontrivial toroidal topology) into account as done in [7].

The behaviour of the difference equation

$$Q(\underline{x}; i)\, f(x_1, \dots, x_i, \dots, x_N) = f(x_1, \dots, x_i + \kappa, \dots, x_N) \qquad i = 1, \dots, N \tag{1.1}$$

where $f(\underline{x})$ is a vector-valued function on N variables x_i, $Q(\underline{x}; i)$ a family of linear operators and κ an arbitrary shift parameter, indeed resembles this problematic nature: the

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operators $Q(\underline{x}; i)$ can be regarded as types of generalized transfer matrices and therefore the analogy with a quantum spin chain becomes obvious.

In section 2 we formulate this equation in a way adapted to quantum symmetry and obtain solutions by a generalized Bethe ansatz. In section 3 they are shown to be highest-weight vectors, and we also calculate their weights. For the sake of transparency both sections are fixed to $U_q[sl(2)]$ containing all essential features of a quantum group. Finally, for completeness, we briefly discuss the aspects of the higher-ranked case in section 4, followed by a summary of the results given in section 5.

2. The generalized Bethe ansatz

Consider N vector spaces $V_i \simeq \mathbb{C}^2$, each given as the representation space of the fundamental representation of $U_q[sl(2)]$. The basis vectors will be denoted by $|1\rangle$ and $|2\rangle$, respectively. The R-matrix then acts as a linear operator on two of such spaces V_i and V_j:

$$R_{ij} : V_i \otimes V_j \to V_j \otimes V_i \tag{2.1}$$

and is given by the quasi-triangular Hopf algebraic structure of $U_q[sl(2)]$ [8]. In the natural basis of tensor products its matrix form reads

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & (1-q^{-2}) & q^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

If in addition one associates a variable x_i with each space V_i, it is possible to define a 'spectral-parameter'-dependent R-matrix:

$$R(x) := \frac{q\mathrm{e}^{x/2}R - q^{-1}\mathrm{e}^{-x/2}PR^{-1}P}{q\mathrm{e}^{x/2} - q^{-1}\mathrm{e}^{-x/2}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(x) & c_-(x) & 0 \\ 0 & c_+(x) & b(x) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

where P is the permutation operator in the sense of (2.1)

$$P_{ij}\,(v_i \otimes v_j) = v_i \otimes v_j \qquad v_{i,j} \in V_{i,j}$$

and $x = x_i - x_j$. The Boltzmann weights read explicitly

$$b(x) = \frac{\mathrm{e}^{x/2} - \mathrm{e}^{-x/2}}{q\mathrm{e}^{x/2} - q^{-1}\mathrm{e}^{-x/2}} \qquad c_\pm(x) = \frac{\mathrm{e}^{\pm x/2}(q - q^{-1})}{q\mathrm{e}^{x/2} - q^{-1}\mathrm{e}^{-x/2}}. \tag{2.4}$$

$R(x)$ satisfies the Yang–Baxter equation

$$R_{12}(x_1 - x_2)R_{13}(x_1 - x_3)R_{23}(x_2 - x_3) = R_{23}(x_2 - x_3)R_{13}(x_1 - x_3)R_{12}(x_1 - x_2). \tag{2.5}$$

One defines a monodromy matrix $T_0(\underline{x}, x_0)$ acting on the tensor product space $V = \bigotimes_{i=1}^N V_i$ and an additional auxiliary space $V_0 \simeq \mathbb{C}^2$

$$T_0(\underline{x}, x_0) := R_{10}(x_1 - x_0)R_{20}(x_2 - x_0) \cdots R_{N0}(x_N - x_0). \tag{2.6}$$

However, as we will see later it is more useful to work with the doubled monodromy matrix as proposed in [9] some years ago as an application of the 'reflection' equation introduced in [10]. In what follows we will use the special type introduced in [7], which is given by

$$\mathcal{T}_0(\underline{x}, x_0) := R_{01}R_{02} \cdots R_{0N}R_{10}(x_1 - x_0)R_{20}(x_2 - x_0) \cdots R_{N0}(x_N - x_0). \tag{2.7}$$

Its dependence on V_0 becomes obvious if $\mathcal{T}_0$ is written as a matrix with respect to the auxiliary space:

$$\mathcal{T}_0 = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}. \tag{2.8}$$

Equation (2.5) implies the Yang–Baxter equation for $\mathcal{T}$:

$$R_{ab}(v-u)\mathcal{T}_a(\underline{x};u)R_{ba}\mathcal{T}_b(\underline{x};v) = \mathcal{T}_b(\underline{x};v)R_{ba}\mathcal{T}_a(\underline{x};u)R_{ba}(v-u) \tag{2.9}$$

giving the following commutation relations for the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$:

$$\begin{aligned}
&\left[\mathcal{B}(\underline{x};u), \mathcal{B}(\underline{x};v)\right] = 0 \\
&\mathcal{A}(\underline{x};u)\mathcal{B}(\underline{x};v) = q^{-1}b^{-1}(u-v)\mathcal{B}(\underline{x};v)\mathcal{A}(\underline{x};u) \\
&\qquad\qquad - \mathcal{B}(\underline{x};u)\left[q^{-1}\frac{c_-(u-v)}{b(u-v)}\mathcal{A}(\underline{x};v) + (1-q^{-2})\mathcal{D}(\underline{x};v)\right] \\
&\mathcal{D}(\underline{x};u)\mathcal{B}(\underline{x};v) = qb^{-1}(v-u)\mathcal{B}(\underline{x};v)\mathcal{A}(\underline{x};u) - q\frac{c_-(v-u)}{b(v-u)}\mathcal{B}(\underline{x};u)\mathcal{D}(\underline{x};v)
\end{aligned} \tag{2.10}$$

(for a detailed proof see [7]).

Analogous to the definition (2.7), consider a further set of monodromy-type matrices defined by

$$\mathcal{T}^Q(\underline{x};i) := R_{01}R_{02}\cdots R_{0N}R_{10}(x_1-x_0)\cdots P_{i0}\cdots R_{N0}(x_N-(x_0+\kappa)) \qquad (i=1,\ldots,N) \tag{2.11}$$

where κ is the arbitrary shift parameter which has already appeared in (1.1). They still have the block structure of (2.8), but no longer depend on the parameter x_0 of the auxiliary space. The monodromy matrices $\mathcal{T}$ and $\mathcal{T}^Q$ now fulfill another Yang–Baxter equation:

$$R_{ba}(x_i-u)\,\mathcal{T}_b(\underline{x}';u)\,R_{ab}\,\mathcal{T}_a^Q(\underline{x};i) = \mathcal{T}_a^Q(\underline{x};i)\,R_{ba}\,\mathcal{T}_b(\underline{x};u)\,R_{ab}(x_i+\kappa-u). \tag{2.12}$$

Again we give some commutation rules relating their matrix elements:

$$\begin{aligned}
&\mathcal{A}^Q(\underline{x};i)\mathcal{B}(\underline{x};u) = q^{-1}b^{-1}(x_i+\kappa-u)\mathcal{B}(\underline{x}';u)\mathcal{A}^Q(\underline{x};i) \\
&\qquad\qquad - \mathcal{B}^Q(\underline{x};i)\left[q^{-1}\frac{c_-(x_i+\kappa-u)}{b(x_i+\kappa-u)}\mathcal{A}(\underline{x};u) + (1-q^{-2})\mathcal{D}(\underline{x};u)\right] \\
&\mathcal{D}^Q(\underline{x};i)\mathcal{B}(\underline{x};u) = qb^{-1}(x_i-u)\mathcal{B}(\underline{x}';u)\mathcal{D}^Q(\underline{x};i) - q\frac{c_-(u-x_i)}{b(u-x_i)}\mathcal{B}^Q(\underline{x};i)\mathcal{D}(\underline{x};u).
\end{aligned} \tag{2.13}$$

The first terms in equations (2.10) and (2.13) are called ‘wanted’, and all others ‘unwanted’. Now taking the Markov trace† over $\mathcal{T}^Q$ gives the operator on the left-hand side of the difference equation (1.1):

$$Q(\underline{x};i) := \mathrm{tr}_q\,\mathcal{T}^Q(\underline{x};i) = \mathcal{A}^Q(\underline{x};i) + q^{-2}\mathcal{D}^Q(\underline{x};i). \tag{2.14}$$

Denote by Ω the usual reference state ($\Omega = |1\rangle^{\otimes N}$) and apply an arbitrary number m of $\mathcal{B}$ operators thereto, defining the following:

Bethe ansatz vector. This is defined as

$$f(\underline{x}) = \sum_{\underline{u}} B(\underline{x};u_m)\cdots B(\underline{x};u_1)\,\Omega\,g(\underline{x};\underline{u}) \tag{2.15}$$

† Its asymmetric form results from the choice of normalization in (2.2).

where the summation over $\underline{u}$ is specified by

$$\sum_{\underline{u}} = \sum_{\substack{l_1\in\mathbb{Z}\\ u_1=\tilde{u}_1+l_1\kappa}} \cdots \sum_{\substack{l_1\in\mathbb{Z}\\ u_m=\tilde{u}_m+l_m\kappa}} \qquad \text{(where } \underline{\tilde{u}} \text{ is an arbitrary set of complex numbers)} \tag{2.16}$$

and the the function $g(\underline{x};\underline{u})$ is defined by

$$g(\underline{x};\underline{u}) = \prod_{i,j} \psi(x_i - u_j) \prod_{k<l} \tau(u_k - u_l). \tag{2.17}$$

Theorem. The difference equation (1.1) defined by (2.14) is solved by the Bethe vectors (2.15) if the functions $\psi(x)$ and $\tau(x)$ satisfy the difference equations

$$q^{-1}b(x+\kappa)\,\psi(x+\kappa) = \psi(x) \qquad q^2\frac{\tau(x)}{b(x)} = \frac{\tau(x-\kappa)}{b(-x+\kappa)}. \tag{2.18}$$

Remark. As a variation of the solutions given in [5] the following functions fulfill this conditions:

$$\psi(x) = \frac{(q^2\mathrm{e}^x;\mathrm{e}^{-\kappa})_\infty}{(\mathrm{e}^x;\mathrm{e}^{-\kappa})_\infty} \qquad \tau(x) = (1-\mathrm{e}^x)\frac{(q^{-2}\mathrm{e}^{x-\kappa};\mathrm{e}^{-\kappa})_\infty}{(q^2\mathrm{e}^x;\mathrm{e}^{-\kappa})_\infty} \tag{2.19}$$

where

$$(z;p)_\infty := \prod_{n=0}^{\infty}(1-zp^n).$$

Proof. We apply the operator $Q(\underline{x};i)$ in its decomposition (2.14) to $f(\underline{x})$. Using relations (2.11) and (2.14) one commutes the operators $\mathcal{A}^Q$ and $\mathcal{D}^Q$ to the right, where they act on the reference state Ω according to

$$\mathcal{A}^Q(\underline{x};i)\Omega = \Omega, \quad \mathcal{D}^Q(\underline{x},i)\Omega = 0$$

and

$$\mathcal{A}(\underline{x};u)\Omega = \Omega \qquad \mathcal{D}(\underline{x},u)\Omega = \prod_{j=1}^{N} b(x_j-u)\,\Omega$$

respectively. The wanted term contribution of $\mathcal{A}^Q$ reads

$$\mathcal{A}(\underline{x};i)\sum_{\underline{u}} \mathcal{B}(\underline{x};u_m)\cdots\mathcal{B}(\underline{x};u_1)\,\Omega\, g(\underline{x},\underline{u}) =$$

$$= \sum_{\underline{u}} \mathcal{B}(\underline{x}';u_m)\cdots\mathcal{B}(\underline{x}';u_1)\,\Omega \prod_{j=1}^{m} q^{-1}b^{-1}(x_i+\kappa-u_j) g(\underline{x},\underline{u}) = f(\underline{x}')$$

where in the last step the quasi-periodic property of ψ (equations (2.18)) has been used. The $q^{-2}\mathcal{D}^Q$ wanted contribution vanishes due to the fact that $\mathcal{D}^Q(\underline{x};i)\,\Omega = 0$.

In a second step one has to verify that all other terms cancel each other under the sum (2.16). Denote the unwanted terms obtained from $\mathcal{A}^Q$ ($q^{-2}\mathcal{D}^Q$) that are proportional to $\mathcal{B}^Q(\underline{x};i)\mathcal{B}(\underline{x};u_{m-1})\cdots\mathcal{B}(\underline{x};u_1)\,\Omega$ by $\mathrm{uw}_{\mathcal{A}}^{(i,j)}$ ($\mathrm{uw}_{\mathcal{D}}^{(i,j)}$). (They result when one first commutes the 'unwanted' terms due to (2.13) and then always the wanted ones due to (2.10).)

$$\mathrm{uw}_{\mathcal{A}}^{(i,m)} = \Bigg[-q^{-1}\frac{c_-(x_i+\kappa-u_m)}{b(x_i+\kappa-u_m)}\prod_{k<m} q^{-1}b^{-1}(u_m-u_k)$$

$$-(1-q^{-2})\prod_{k<m} qb^{-1}(u_k-u_m)\prod_{j=1}^{N} q^{-1}b(x_j-u_m)\Bigg]$$

$$\times\, \mathcal{B}^{\mathcal{Q}}(\underline{x};i)\mathcal{B}(\underline{x};u_{m-1})\cdots\mathcal{B}(\underline{x};u_1)\,\Omega g(\underline{x};\underline{u}) \tag{2.20}$$

$$\mathrm{uw}_{\mathcal{D}}^{(i,m)} = -q^{-1}\frac{c_-(u_m-x_i)}{b(u_m-x_i)}\prod_{k<m} q\, b^{-1}(u_k-u_m)$$

$$\times\prod_{j=1}^{N} q^{-1} b(x_i-u_m)\mathcal{B}^{\mathcal{Q}}(\underline{x};i)\mathcal{B}(\underline{x};u_{m-1})\cdots\mathcal{B}(\underline{x};u_1)\,\Omega g(\underline{x};\underline{u}). \tag{2.21}$$

Using the symmetry property $(c_-/b)(-x) = -(c_-/b)(x) - (1-q^{-2})$ combine (2.21) and the second term of (2.20). Then both expressions in (2.18) are applied to this term, and obviously this term cancels with the first one of (2.20) under the sum (2.16), which completes the proof. □

3. Bethe vectors and highest-weight modules

The generators of $U_q[sl(2)]$ can be derived from the monodromy matrix $T_0(\underline{x};u)$ (2.7) in the limits $u\to\pm\infty$:

$$\begin{aligned} T &= \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22}\end{pmatrix} := \lim_{u\to-\infty} T_0(\underline{x};u) = q^{-N}\begin{pmatrix} 1 & 0 \\ (q-q^{-1})J_+ & 1\end{pmatrix} q^{\mathbf{W}} \\ \tilde{T} &:= \lim_{u\to+\infty} T_0(\underline{x};u) = q^{N} q^{-\mathbf{W}}\begin{pmatrix} 1 & (q-q^{-1})J_- \\ 0 & 1\end{pmatrix}\end{aligned} \tag{3.1}$$

where $\mathbf{W} = \mathrm{diag}\{W_1, W_2\}$ contains the Cartan elements. In order to prove the highest-weight property of $f(\underline{x})$, i.e. the statement

$$T^{21} f(\underline{x}) \propto J_+ f(\underline{x}) = 0 \tag{3.2}$$

analogous to (3.1) we introduce

$$\mathcal{T} := \tilde{T}^{-1}\, T \tag{3.3}$$

as a limit of $\mathcal{T}(\underline{x};u)$. First we show that $\mathcal{T}^{21} f(\underline{x}) = 0$. The Yang–Baxter equation (2.12) implies

$$\left[\mathcal{T}^{21}, \mathcal{B}(u)\right] = (1-q^{-2})\left[\mathcal{A}(u)\mathcal{T}^{22} - \mathcal{T}^{22}\mathcal{D}(u)\right]. \tag{3.4}$$

Again due to the commutativity of the $\mathcal{B}$-operators it is sufficient to consider the term proportional to $\mathcal{B}(u_m)\cdots\mathcal{B}(u_2)$. Because $\mathcal{A}(u)$, $\mathcal{D}(u)$ and $\mathcal{T}^{22}$ act diagonally on Ω it remains for us to show that

$$\sum_{\underline{u}} [\mathcal{A}(u_1) - \mathcal{D}(u_1)]\,\Omega\, g(\underline{x};\underline{u}) = 0$$

which follows directly from (2.18). Since $\tilde{T}^{-1}$ is an invertible operator, equation (3.3) implies the statement (3.2).

The weights ω of the Bethe vectors $f(\underline{x})$ are defined by

$$q^{\mathbf{W}_i} f(\underline{x}) = q^{\boldsymbol{\omega}_i} f(\underline{x}) \qquad \boldsymbol{\omega} = (\omega_1, \omega_2).$$

The commutation relations

$$\mathcal{A}\mathcal{B}(u) = q^{-2}\mathcal{B}(u)\mathcal{A} \qquad \mathcal{D}\mathcal{B}(u) = q^{2}\mathcal{B}(u)\mathcal{D}$$

and equation (3.3) therefore imply

$$T^{11} f(\underline{x}) = q^{N-m} f(\underline{x}) \qquad \text{and} \qquad T^{22} f(\underline{x}) = q^{m} f(\underline{x})$$

giving the weight vector $\boldsymbol{\omega} = (N-m, m)$ as expected.

4. The higher-ranked case $U_q[sl(n)]$

In this section we briefly discuss the case of a $U_q[sl(n)]$ difference equation. (For a more detailed description of the nested Bethe ansatz method in general we refer the reader to [1, 7].)

Denote by E_{ij} the unit matrices in $M_{n,n}(\mathbb{C})$. The $U_q[sl(n)]$ R-matrix is then given by

$$R = \sum_i E_{ii} \otimes E_{ii} + q^{-1} \sum_{i \neq j} E_{ii} \otimes E_{jj} + (1 - q^{-2}) \sum_{i > j} E_{ij} \otimes E_{ji} \qquad (4.1)$$

whereas the definitions for $R(x)$, $T_0(\underline{x}; u)$ and $\mathcal{T}_0(\underline{x}; u)$ can be carried over directly from the equations (2.3), (2.6) and (2.7). The latter two operators are now considered as $n \times n$ matrices; the commutation relations of their elements read in analogy with equations (2.10) and (2.13):

$$\mathcal{A}(\underline{x}; u)\mathcal{B}_\gamma(\underline{x}; v) = q^{-1}b^{-1}(u - v)\mathcal{B}_\gamma(\underline{x}; v)\mathcal{A}(\underline{x}; u)$$
$$- q^{-1}\left[\frac{c_-(u - v)}{b(u - v)}\mathcal{B}_\gamma(\underline{x}; u)\mathcal{A}(\underline{x}; v) + (q - q^{-1})\mathcal{B}_\alpha(\underline{x}; u)\mathcal{D}_{\alpha\gamma}(\underline{x}; v)\right]$$

$$\mathcal{D}_{\beta\gamma}(\underline{x}; u)\mathcal{B}_\delta(\underline{x}; v) = q\, b^{-1}(v - u)\Big[\mathcal{B}_{\gamma''}(\underline{x}; v)\mathcal{D}_{\beta'\delta'}(\underline{x}; u)R^{\delta'\gamma'}_{\delta\gamma}(v - u)R^{\gamma''\beta}_{\beta'\gamma'}$$
$$- c_-(v - u)R^{\gamma'\beta}_{\beta'\gamma}\mathcal{B}_{\gamma'}(\underline{x}; u)\mathcal{D}_{\beta'\delta}(\underline{x}; v)\Big]$$

$$\mathcal{A}^Q(\underline{x}; i)\mathcal{B}_\gamma(\underline{x}; u) = q^{-1}b^{-1}(x_i + \kappa - u)\mathcal{B}_\gamma(\underline{x}'; u)\mathcal{A}^Q(\underline{x}; i)$$
$$- q^{-1}\left[\frac{c_-(x_i + \kappa - u)}{b(x_i + \kappa - u)}\mathcal{B}^Q_\gamma(\underline{x}; i)\mathcal{A}(\underline{x}; u) + (q - q^{-1})\mathcal{B}^Q_\alpha(\underline{x}; i)\mathcal{D}_{\alpha\gamma}(\underline{x}; u)\right]$$

$$\mathcal{D}^Q_{\beta\gamma}(\underline{x}; i)\mathcal{B}_\delta(\underline{x}; u) = q\, b^{-1}(u - x_i)\Big[\mathcal{B}_{\gamma''}(\underline{x}'; u)\mathcal{D}^Q_{\beta'\delta'}(\underline{x}; i)R^{\delta'\gamma'}_{\delta\gamma}(u - x_i)R^{\gamma''\beta}_{\beta'\gamma'}$$
$$- c_-(u - x_i)R^{\gamma'\beta}_{\beta'\gamma}\mathcal{B}^Q_{\gamma'}(\underline{x}; i)\mathcal{D}_{\beta'\delta}(\underline{x}; u)\Big]$$

where the lower-case greek indices run from 2 to n. The operators $Q(\underline{x}; i)$, which define (1.1) are given by the $U_q[sl(n)]$ Markov trace

$$Q(\underline{x}; i) := \mathrm{tr}_q \mathcal{T}^Q(\underline{x}; i) = \mathcal{A}^Q(\underline{x}; i) + \sum_{\alpha=2}^{n} q^{-2(\alpha-1)}\mathcal{D}^Q_{\alpha\alpha}(\underline{x}; i). \qquad (4.2)$$

We further denote the number of particles by N_n. The Bethe vectors that solve (1.1) are created by the action of N_{n-1} $\mathcal{B}$-operators and read

$$f(\underline{x}) = \sum_{\underline{u}} \mathcal{B}_{\beta_{N_{n-1}}}(\underline{x}; u_{N_{n-1}}) \cdots \mathcal{B}_{\beta_1}(\underline{x}; u_1)\, \Omega\, g^{\underline{\beta}}(\underline{x}, \underline{u}) \qquad (4.3)$$

where, in contrast to section 2, $g(\underline{x}; \underline{u})$ is a function with values in $V^{(n-1)} = \otimes^{N_{n-1}}\mathbb{C}^{(n-1)}$ given by the ansatz

$$g(\underline{x}; \underline{u}) = \prod_{i,j} \psi(x_i - u_j) \prod_{k<l} \tau(u_k - u_l)\, f^{(n-1)}(\underline{u}) \qquad (4.4)$$

with functions $\psi(x)$ and $\tau(x)$ as given by (2.19) and a (yet undetermined) function $f^{(n-1)}$ with values in $V^{(n-1)}$. To prove (1.1) one has to apply $Q(\underline{x}; i)$ to $f(\underline{x})$; the 'wanted' contribution of $\mathcal{A}^Q$ again produces the right-hand side of (1.1). On the other hand the 'unwanted' terms cancel exactly if $f^{(n-1)}$ satisfies the $(n-1)$-dimensional analogue of (1.1). Therefore we repeat the ansatz (4.3) for $f^{(n-1)}$ and all the resulting subsequent Bethe ansatz levels, where consequently the number of $\mathcal{B}$-operators used at the kth level is denoted by N_{n-k}. Finally after $n-2$ steps the problem has been reduced to the $U_q[sl(2)]$ problem already solved in section 2.

The highest-weight property of the Bethe vectors (4.3) is proved in a way parallel to that followed in section 3. At some stages the higher-ranked case is a little more involved, but those aspects have been already treated carefully in [13].

The resulting weight vector $\boldsymbol{\omega}$ then reads

$$\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) = (N_n - N_{n-1}, \ldots, N_2 - N_1, N_1) \tag{4.5}$$

again fulfilling the maximum weight condition

$$\omega_1 \geqslant \cdots \geqslant \omega_n \geqslant 0. \tag{4.6}$$

5. Conclusions

Starting from the $U_q[sl(2)]$ R-matrix we derived a family of q-deformed discrete Knizhnik–Zamolodchikov equations. These difference equations are investigated in an analogous way to that used for the problem of relating a XXZ-type spin-chain to its corresponding quantum group: the quantum symmetry is preserved only by a suitable choice of boundary conditions. Using the the generalization of the algebraic Bethe ansatz as developed in [1] we constructed solutions to this difference equations. These solutions are then proved to be highest-weight vectors of a representation of $U_q[sl(2)]$, the underlying quantum group structure. Using the variant of the nested Bethe ansatz method we extended the results to the higher-ranked symmetry of $U_q[sl(n)]$.

In a further paper we will apply these results to the form-factor problem of the quantum Sine–Gordon model in an way analogous to that used in [2] for the Sine–Gordon model. The main use of the method presented here will lie in a better classification of the derived form-factors due to the heighest weight property of the states.

On the other hand, it is possible to extend the method to other types of symmetries, e.g. quantum groups based on graded Lie algebras (see, e.g., [14]). Further generalizations of the nested Bethe ansatz methods for other simply laced Lie algebras and their q-deformed analogues (e.g., the $O(2n)$ case, for the conventional algebraic Bethe ansatz see [15]) are also under consideration.

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